Determine the convergence of the following series:

(1) $\displaystyle\sum_{n=1}^{\infty} \underbrace{\frac{n^2+2n+1}{n^4+n^2+2n+1}}$

for large n, this looks like $\frac{1}{n^2}$, Choose $\sum b_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2+2n+1}{n^4+n^2+2n+1} \cdot \frac{n^2}{1} = \lim_{n\to\infty} \frac{n^4+2n^3+1}{n^4+n^2+2n+1} = 1$$

So by LCT, since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, our series converges

(2) $\displaystyle\sum_{n=1}^{\infty} \sin\left(\frac{2\pi}{n}\right)$ We know $\lim_{x\to\infty} x\sin\left(\frac{1}{x}\right) = 1$, so mimic this.

Choose $\sum b_n = \sum_{n=1}^{\infty} \frac{2\pi}{n}$. Then $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n}{2\pi} \sin\left(\frac{2\pi}{n}\right) = 1$

& since $\sum_{n=1}^{\infty} \frac{2\pi}{n}$ diverges, by the LCT, so does $\sum_{n=1}^{\infty} \sin\left(\frac{2\pi}{n}\right)$.

(3) $\displaystyle\sum_{n=1}^{\infty} \underbrace{\frac{n+1}{n\sqrt{n}}}$

for large n, looks like $\frac{1}{\sqrt{n}}$. Choose $\sum b_n = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n\sqrt{n}+\sqrt{n}}{n\sqrt{n}} = 1.$$ Since $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges,

so does $\sum_{n=1}^{\infty} \frac{n+1}{n\sqrt{n}}$.

(4) $\sum_{n=1}^{\infty} \frac{1}{n^{1+\frac{1}{n}}}$

Recall $\lim_{n\to\infty} n^{1/n} = 1$ (use ln & l'Hôpital's rule to get this)

Choose $\sum b_n = \sum_{n=1}^{\infty} \frac{1}{n}$. $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n}{n^{1+\frac{1}{n}}} = \lim_{n\to\infty} \frac{1}{n^{1/n}} = 1$

So, since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, so does $\sum_{n=1}^{\infty} \frac{1}{n^{1+\frac{1}{n}}}$ by LCT.

(5) $\sum_{n=1}^{\infty} \frac{n!}{n^n}$

$$\frac{n!}{n^n} = \frac{1\cdot 2\cdot 3\cdot \cdots \cdot (n-1)\cdot n}{n\cdot n\cdot n\cdot \cdots \cdot n \cdot n} = \frac{1}{n}\cdot\frac{2}{n}\cdot\frac{3}{n}\cdot \cdots \cdot \frac{n-1}{n}\cdot\frac{n}{n}$$
$$\leq \frac{1}{n}\cdot\frac{2}{n}\cdot 1\cdot \cdots \cdot 1 \cdot 1 = \frac{2}{n^2}$$

So, $\sum_{n=1}^{\infty} \frac{n!}{n^n} \leq \sum_{n=1}^{\infty} \frac{2}{n^2} < \infty$ by p-series test.

So, $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ converges

(6) $\sum_{n=4}^{\infty} \frac{\arctan 2n}{n^{\pi}}$

Recall $-\frac{\pi}{4} < \arctan x < \frac{\pi}{4}$ & $\arctan x > 0$, $x > 0$

so, $0 < \arctan 2n < \frac{\pi}{4}$, $n \geq 4$.

$\sum_{n=4}^{\infty} \frac{\arctan 2n}{n^{\pi}} \leq \sum_{n=4}^{\infty} \frac{\left(\frac{\pi}{4}\right)}{n^{\pi}} < \infty$ converges by p-test since $p = \pi > 1$.

Thus, $\sum_{n=4}^{\infty} \frac{\arctan 2n}{n^{\pi}}$ converges